# INVERSE PROBLEM ON ISOMORPHISM THEOREM OF A^{p}(G)$-ALGEBRAS ${1}le{p}le{2} (Nonlinear Analysis and Convex Analysis)



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# INVERSE PROBLEM ON ISOMORPHISM THEOREM OF $A^p(G)$-ALGEBRAS $1 \leq p \leq 2$

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## Abstract

Let $G_1$ and $G_2$ be locally compact Hausdorff groups, $E(G_1)$ and $E(G_2)$ the function spaces (Banach algebras or Banach spaces ) on $G_1$ and $G_2$ respectively. Then it is known that if $G_1 \simeq G_2$, implies $E(G_1)$ and $E(G_2)$ are isomorphic. Naturally, an inverse problem arises that

(P) Whether an algebaic isomorphism $\Phi : E(G_1) \longrightarrow E(G_2)$ could deduce $G_1 \simeq G_2$?

In this paper, we would solve Problem (P) for $A^p(G)-$algebras, $1 \leq p \leq 2$.

## 1. Preliminaries

(1) 1948, Y. Kawada [7] solved this problem under bipositive isomorphism $\Phi : L^1(G_1) \longrightarrow L^1(G_2)$.

(2) 1952, Wendel [13] proved (P) under the isomorphism $\Phi$ from the algebra $L^1(G_1)$ onto $L^1(G_2)$ by assuming $\Phi$ is a norm nonincreasing.

(3) 1965, Edwards [2] considered the groups $G_i(i = 1, 2)$ are compact, and if there exists a bipositive isomorphism of $L^p(G_1)$ onto $L^p(G_2)$ to get ,then $G_1 \simeq G_2$. He asked whether the compact groups $G_1$ and $G_2$ are necessarily homeomorphic, if bipositive is replaced by isometry?

(4) 1966, The affirmative answer to this question in [2] by positive replaced isometry was given by Strichartz [12].

(5) 1968, Further, Parrott [11] proved the question in Edwards [2] for general locally compact groups $G_1$ and $G_2$ if there is an isomertic transformation of $L^p(G_1)$ onto $L^p(G_2)$ $(1 \leq p < \infty,\ p \neq 2)$.

Remark : The Lebesgue space $L^p(G)$ need not be an algebra if $G$ is not compact.

(6) 1973, Lai/Lien [10] solved the problem (P) by assume that if there exists an injective bipositive linear mapping from the Banach space $L^p(G_1)$ onto Banach space $L^p(G_2)$, then $G_1 \simeq G_2$ is deduced.

(7) Some other isomorphism problems were solved by Johnson [6], Gaudry [4] and Figa Talamanca [3] in different view points.

(8) In this article we would solve problem (P) on the Banach algebra $A^p(G)$, $1 \le p \le 2$.

## 2. $A^p(G)-$ALGEBRAS, $1 \le p < \infty$

In this paper, we would consider the isomorphism theorem for $A^p(G) - algebras$.

Let $G$ be a $LCA$ group with dual group $\widehat{G}$. The space $A^p(G)$ is defined by

$$A^p(G) = \{f \in L^1(G)\ ;\ \text{Fourier transform } \widehat{f} \in L^p(\widehat{G})\}, 1 \le p < \infty. \tag{1}$$

Then $A^p(G)$ is a commutative Banach algebra under convolution product with the norm given by

$$\| f \|^p = \| f \|_1 + \| \widehat{f} \|_p, \text{ for each } \quad p,\ 1 \le p < \infty \text{ for } f \in A^p(G). \tag{2}$$

The norm $\| \ \|^p$ is equivalent to $\max(\| f \|_1, \| \widehat{f} \|_p)$.

Since $f \in A^p(G) \Longrightarrow \widehat{f} \in L^P(\widehat{G}) \cap C_0(\widehat{G})$, thus $\widehat{f} \in L^r(\widehat{G})$ for $r > p > 1$, but such $f \notin A^r$ for $1 \le p \le 2 \le q < r < \infty$.

By this fact, we know that $A^p(G)$ can not include all Fourier transforms of $C_c(G) \cap A^r$. And $A^1(G) \supset A^p(G) \supset A^2(G) \supset A^q(G) \supset C_0(G)$, where $A^1(G) = \bigcup_{1<p\le 2} A^p(G)$ is the closure of such union sets.

We then conclude that

$$1 \le p \le 2,\ C_c \cap A^p(G) \text{ is dense in } A^p(G) \text{ with respect to the } A^p\text{-norm.}$$

Thus,

$$\text{if } f \in A^p(G), \text{ then } \widehat{f} \in L^p(\widehat{G}) \text{ and } \widehat{f} \in L^q(\widehat{G}) \text{ for } p \le 2 < q,\ f \notin A^q(G),$$

$$\text{and so } \forall p,\ 1 \le p \le 2 \le q < \infty,\ \tfrac{1}{p} + \tfrac{1}{q} = 1,\ A^p(G) \cap A^q(G) = \emptyset.$$

Hence the index $p$, only taken in the interval $1 \le p \le 2$ could get $T(A^p) \subset A^p$ by a continuous linear operator $T$. So we can discuss the multipliers $T$ on $A^p(G)$ only taken

$1 \leq p \leq 2$ which could get $T(A^p) \subset A^p$. Therefore in later part, all $A^p(G)$ we discuss will take $1 \leq p \leq 2$.

## 3. Multipliers of $A^p(G)$

A multiplier $T$ of $A^p(G)$ is a continuous linear mapping of $A^p(G), 1 \leq p \leq 2$ into itself, such that

$$T(f * g) = T(f) * g = f * T(g), \text{ for all } f, g \in A^p(G).$$

In order to solve problem (P) on $A^p(G)-$algebras. We use a technique by passing the multiplier of $A^p(G)$, thus we subscrip the definition of $A^p(G)$, as follows. Let $\mathfrak{L}(A^p)$ be the space of all bounded linear operator of $A^p(G)$, $1 \leq p \leq 2$ into itself.

**Definition 1.** *An operator $T \in \mathfrak{L}(A^p(G))$ is said to be a multiplier of $A^p(G)$ if*

$$T(f * g) = Tf * g = f * Tg \text{ for } f, g \in A^p(G). \tag{3}$$

The concept of multiplier $T$, one can consult Lai/ Lee / Liu [9, Theorem 1.1]. It deduces the space $\mathfrak{M}(A^p)$ of multipliers of $A^p(G)$ is isometrically isomorphic to $M(G)$, the space of all regular measures of $G$, that is

$$\mathfrak{M}(A^p) \cong \mathfrak{M}(L^1) \cong M(G), \ 1 \leq p \leq 2. \tag{4}$$

On the other hand, it is known that $A^p(G)$ is essential $L^1(G)$-module, since $L^1(G)$ has bounded approximate identity of norm 1 [9, Theorem 2.1]. It is remarkable that $A^p(G)$ has no $A^p$-uniform bounded approximate identity [8, p.574].

$$A^p * L^1 = A^p, \text{ and } ||f * g||^p \leq ||f||^p ||g||, \text{ for } f \in A^p, g \in L^1. \tag{5}$$

Thus the space $\mathfrak{M}(A^p, L^1)$ of multiplier $A^p$ into $L^1$ is identical to $\mathfrak{M}(A^p)$. Hence there exists a unique $\mu \in M(G)$ such that

$$Tf = \mu * f \text{ for all } f \in A^p(G) \tag{6}$$

for any $T \in \mathfrak{M}(A^p, L^1) \cong \mathfrak{M}(A^p)$. By the property of $A^p(G)-$algebras, we will show the Isomorphism Theorem of $A^p(G)-$algebras can be stated as the following:

**Theorem 2.** *Let $G_1$ and $G_2$ be locally compact abelian groups and $\Phi$ an algebaic isomorphism of $A^P(G_1)$ onto $A^p(G_2)$, $1 \leq p \leq 2$. Suppose that one of $\widehat{G_1}$ and $\widehat{G_2}$ is connected, then $\Phi$ induces a topological isomorphism $\tau$ carrying $G_2$ onto $G_1$. Furthermore,*

$$\Phi f(x) = c\widehat{x}(x) f(\tau x) \text{ for } f \in A^p(G_1), \text{ and } x \in G_2,$$

*where $\widehat{x}(x)$ is a fixed character on $G_2$ and $c$ a constant depending only on the choice of Haar measure in $G_2$.*

<u>Outline</u> of the proof for the main <u>Theorem</u> is given as follows:

Since the isomorphism

$$\begin{aligned}
&\Phi : A^p(G_1) \overset{onto}{\longrightarrow} A^p(G_2),\\
\Longrightarrow\ &\Phi \text{ maps the } \underline{\text{Maximal ideal}} \text{ spaces } \mathfrak{Max}(A^p(G_1)) \text{ of } A^p(G_1)\\
&\text{on to } \mathfrak{Max}(A^p(G_2)) \text{ of } A^p(G_2),\\
\Longrightarrow\ &\Phi : \mathfrak{Max}(A^p(G_1)) \longrightarrow \mathfrak{Max}(A^p(G_2))\\
&\qquad\qquad \| \qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad (7)\\
\Longrightarrow\ &\Phi : \qquad \widehat{G_1} \quad \overset{\text{onto}}{\longrightarrow} \quad \widehat{G_2}
\end{aligned}$$

The reason of (7) is that since $A^p(G)$ is a semisimple commutative Banach algebra, then the space $\mathfrak{Max}(A^p(G))$ is characterized by $\widehat{G}$.

Hence if one of $\widehat{G_1}$ and $\widehat{G_2}$ is connected, then both of $\widehat{G_1}$ and $\widehat{G_2}$ are <u>connected</u>. Therefore $G_1$ and $G_2$ are non-compact.

Since the theorem in [3] is applicable, we note that operator $T$ commutes with convolution on $A^p(G_1)$ is represented uniquely by $\mu \in M(G_1)$

$$\begin{aligned}
&Tf = \mu * f = 0 \text{ for all } f \in A^p(G_1)\\
&\Longrightarrow \mu = 0.
\end{aligned} \tag{8}$$

Thus we take $\nu \in M(G_2)$ for any $f \in A^p(G_1)$, it can define this operator

$$T : A^p(G_1) \longrightarrow A^p(G_2)$$

by

$$\mu * f = \Phi^{-1}(\nu * \Phi f) = Tf. \tag{9}$$

It is well-defined by (8) since $A^p(G_1)$ is semisimple, and by Loomis [book : p.76 Theorem], one sees that $\Phi$ is bicontinuous and hence $T$ is a multiplier of $A^p(G_1)$, thus $\exists!\ \mu \in M(G_1)$ such that

$$\mu * f = \Phi^{-1}(\nu * \Phi f) = Tf.$$

This $\mu$ is uniquely determined by $\nu$, we define a mapping $\Psi$ of $M(G_2)$ into $M(G_1)$ by

$$\Psi\nu * f = \Phi^{-1}(\nu * \Phi f).$$

It is not hard to prove that $\Psi$ is an isomorphism of $M(G_2)$ onto $M(G_1)$. Since both measure algebras $M(G_1)$ and $M(G_2)$ are semi-simple and commutative, $\Psi$ is bicontinuous and one can show that

$$\Psi|_{A^p(G_2)} \text{ on the algebra } A^p(G_2) \text{ is dense in } L^1(G_2),$$

hence $\Psi|L^1(G_2)$ becomes an isomorphism of $L^1(G_2)$ onto $L^1(G_1)$ [See Rudin's book Theorem 6.6.4]. Hence by Helsen [5], the theorem is complete. □

The full paper about Isomorphism Theorem of $A^p(G)$-algebras will appear in elsewhere.

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